A Note on the Product of Complementary Principal Minors of a Positive Definite Matrix

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ABSTRACT

An upper bound is given for the product of complementary principal minors of a positive definite matrix in terms of its eigenvalues.

Let H be an $n\times n$ positive definite matrix, (i.e., H has real entries, $H^T=H$ and all the eigenvalues of H are positive). Let the eigenvalues of H be $\alpha_1,\alpha_2,\dots,\alpha_n$, where $\alpha_1\geqslant\alpha_2\geqslant\cdots\geqslant\alpha_n$. Let r be an integer with $1\leqslant r\leqslant n-1$, and let H be partitioned in the form

$$H=\begin{bmatrix} H_1 & H_2 \\ H_2^T & H_3 \end{bmatrix},$$

where H_3 is $r\times r$.

In this situation, a result of E. Fischer [3] gives a *lower* bound for the value of the product $\det H_1\cdot\det H_3$, namely,

$$\det H_1\cdot\det H_3\geqslant\det H,$$

or equivalently, in terms of the eigenvalues of H,

$$\det H_1\cdot\det H_3\geqslant\alpha_1\alpha_2\cdots\alpha_n.$$

The purpose of this note is to give an *upper* bound for $\det H_1\cdot\det H_3$ in terms of the eigenvalues of H. The proof makes use of an inequality in the theory of least squares estimation proved by Bloomfield and Watson [2] and by Knott [4].

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First we quote the following lemma with the intention of applying it to a positive definite matrix.

LEMMA. *Let A be a nonsingular $n \times n$ matrix. Let r be an integer with $1 \leqslant r \leqslant n-1$. Let A and A^{-1} be partitioned as*

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix},$$

where A_4 and B_4 are both $r \times r$. Suppose that A_1 is nonsingular.
It then follows that

$$\det A_1 = \det A \cdot \det B_4. \tag{1}$$

Proof. See Aitken [1, p. 99]. ∎

The following is the main result.

THEOREM. *Let H be a positive definite $n \times n$ matrix with eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$, where $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$. Let r be an integer with $1 \leqslant r \leqslant n-1$. Let H be partitioned as*

$$H = \begin{bmatrix} H_1 & H_2 \\ H_2^T & H_3 \end{bmatrix},$$

where H_3 is $r \times r$. Let $q = \min(r, n-r)$. Then

$$\det H_1 \cdot \det H_3 \leqslant \alpha_{q+1}\alpha_{q+2} \cdots \alpha_{n-q} \prod_{k=1}^{q} \left(\frac{\alpha_k + \alpha_{n-k+1}}{2} \right)^2. \tag{2}$$

Proof. Let $D = \operatorname{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$. It then follows that there exists a real orthogonal matrix U such that $UHU^T = D$, so that $H = U^T DU$. Furthermore we can partition U as $[U_1 \;\; U_2]$, where U_1 is $n \times (n-r)$ and U_2 is $n \times r$, with the result that

$$H = \begin{bmatrix} H_1 & H_2 \\ H_2^T & H_3 \end{bmatrix} = \begin{bmatrix} U_1^T D U_1 & U_1^T D U_2 \\ U_2^T D U_1 & U_2^T D U_2 \end{bmatrix}.$$

Also, notice that $H^{-1} = U^T D^{-1} U$, with the result that

$$H^{-1} = \begin{bmatrix} U_1^T D^{-1} U_1 & U_1^T D^{-1} U_2 \\ U_2^T D^{-1} U_1 & U_2^T D^{-1} U_2 \end{bmatrix}. \tag{3}$$

Then,

$$\begin{aligned} \det H_1 \cdot \det H_3 &= \det U_1^T D U_1 \cdot \det U_2^T D U_2 \\ &= \det H \cdot \det U_2^T D^{-1} U_2 \cdot \det U_2^T D U_2 \qquad \text{[by (1) and (3)]} \\ &= \alpha_1 \alpha_2 \cdots \alpha_n \det U_2^T D^{-1} U_2 \cdot \det U_2^T D U_2 \\ &\leqslant \alpha_1 \alpha_2 \cdots \alpha_n \prod_{k=1}^{q} \frac{(\alpha_k + \alpha_{n-k+1})^2}{4 \alpha_k \alpha_{n-k+1}}, \end{aligned}$$

[where $q = \min(r, n-r)$, on using the result of Knott [4, p. 129]]

$$= \alpha_{q+1} \alpha_{q+2} \cdots \alpha_{n-q} \prod_{k=1}^{q} \left(\frac{\alpha_k + \alpha_{n-k+1}}{2} \right)^2,$$

as required. ■

Remark. Equality is possible in (2) for each admissible pair of values of n and r. For example, if $n = 6$ and $r = 2$, take H to be the matrix

$$\begin{bmatrix} \frac{1}{2}(\alpha_1 + \alpha_6) & 0 & 0 & 0 & \frac{1}{2}(\alpha_1 - \alpha_6) & 0 \\ 0 & \frac{1}{2}(\alpha_2 + \alpha_5) & 0 & 0 & 0 & \frac{1}{2}(\alpha_2 - \alpha_5) \\ 0 & 0 & \alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_4 & 0 & 0 \\ \frac{1}{2}(\alpha_1 - \alpha_6) & 0 & 0 & 0 & \frac{1}{2}(\alpha_1 + \alpha_6) & 0 \\ 0 & \frac{1}{2}(\alpha_2 - \alpha_5) & 0 & 0 & 0 & \frac{1}{2}(\alpha_2 + \alpha_5) \end{bmatrix}.$$

This matrix is positive definite, it has $\alpha_1, \alpha_2, \ldots, \alpha_n$ as its eigenvalues, and

$$\det H_1 \cdot \det H_3 = \alpha_3 \alpha_4 \left(\frac{\alpha_1 + \alpha_6}{2} \right)^2 \left(\frac{\alpha_2 + \alpha_5}{2} \right)^2.$$

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REFERENCES

1 A. C. Aitken, *Determinants and Matrices*, Oliver and Boyd, 1959.
2 P. Bloomfield and G. S. Watson, The inefficiency of least squares, *Biometrika* 62:121–128 (1975).
3 E. Fischer, Über den Hadamardschen Determinantensatz, *Arch. Math. Phys. (3)* 13:32–40 (1908).
4 M. Knott, On the minimum efficiency of least squares, *Biometrika* 62:129–132 (1975).

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